Efficient High Order Central Schemes for Multi-Dimensional Hamilton-Jacobi Equations

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Outline

- Introduction
- 1st and 2nd order methods
- High-order methods
- Conclusions

Hamilton-Jacobi Equations

- Equations of the form $\varphi_t + H(\varphi_x) = 0$
  - Where we assume H is at least continuous
  - Evolves discontinuous derivatives even from smooth initial data
    - Viscosity Solution (Crandall, Lions, Evans)
- Applications in control theory, optics, …
  - Encounter high-dimensional spaces

Numerical Methods for HJ

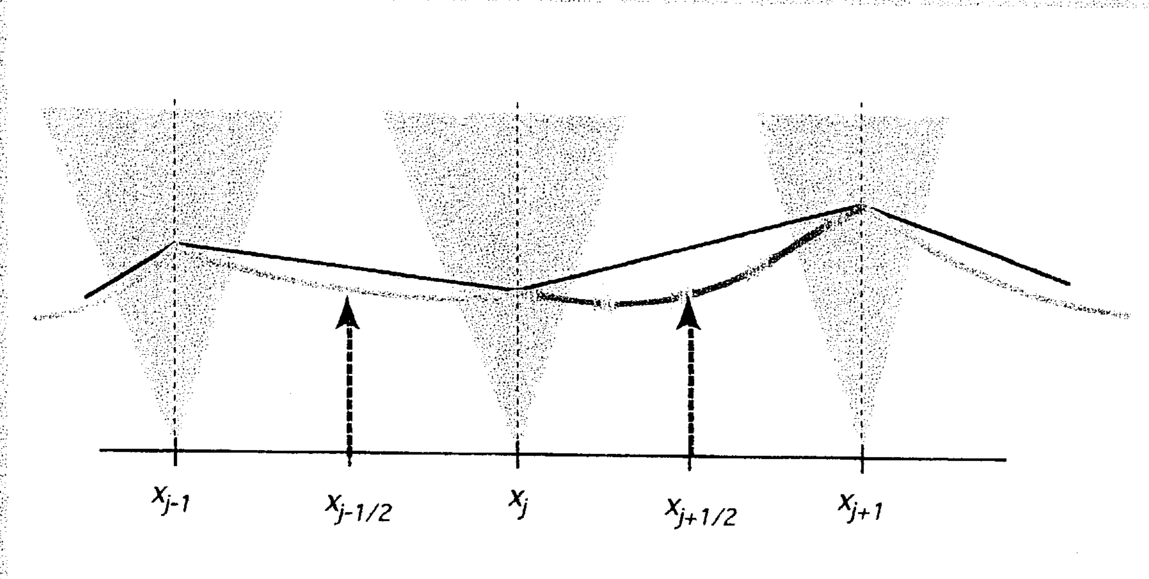
- Numerical Methods for HJ Eqns
  - Complicated by non-smoothness of solutions
    - Known to converge to viscosity solution (Souganitas)
    - Adapt techniques from conservation laws
      - Flux limiters, WENO, Central methods
  - Our Goal: *high-order, efficient, central methods that scale well to high dimension*

Existing Work

- Upwind Schemes
  - Osher and Shu - high-order ENO methods
  - Jiang and Peng - high-order WENO methods
- Central Schemes
  - Lin and Tadmor - 1st and 2nd order staggered
    - Minmod flux limiter on 1st derivative
    - Proved 1st order convergence
  - Kurganov and Tadmor - 1st and 2nd order semi-discrete
    - Minmod limiter on 2nd derivative
    - reduce dissapation by estimating local speed of propagation

The Central Philosophy

- Evolve where data is smooth



- Steps: *reconstruct, evolve, reproject*
- Avoid solving Riemann problems
  - Good for systems and high dimensions

First and Second Order

- Limit the second derivatives and reproject onto original grid points
  - Based on Lin-Tadmor and Kurganov-Tadmor
  - Same work as Lin-Tadmor in 2D
- Evolve at evolution points using quadrature
  - 1st-order method:

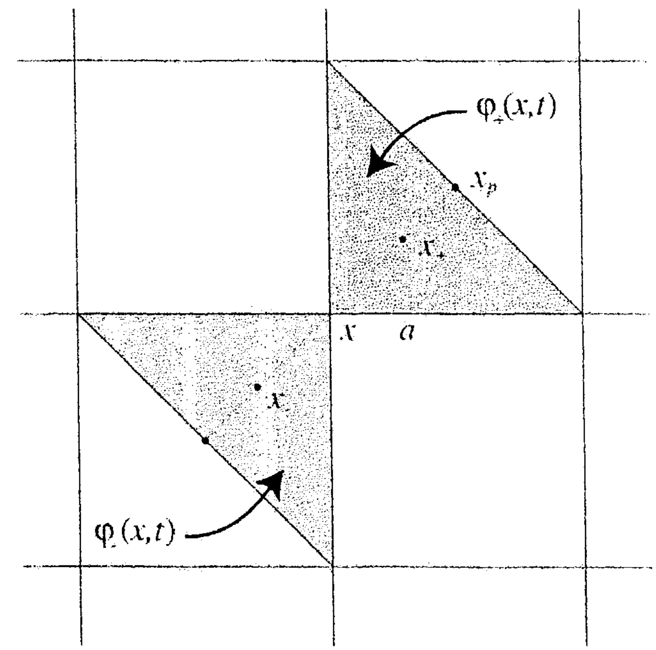
$$\varphi^{m+1} = \varphi^{m} + \frac{1}{4}\left((\Delta\varphi)^{m}_{i+\frac{1}{2}} - (\Delta\varphi)^{m}_{i-\frac{1}{2}}\right) - \frac{\Delta t}{2}\left[H\left(\frac{(\Delta\varphi)^{m}_{i+\frac{1}{2}}}{\Delta x}\right) + H\left(\frac{(\Delta\varphi)^{m}_{i-\frac{1}{2}}}{\Delta x}\right)\right]$$

  - Use Taylor expansion for mid-values in 2nd-order midpoint quadrature
    - Assumes $H \in C^1$

Evolution in R^n

- Partition space into simplices along + and - diagonal
  - Singularities along simplex boundaries
- Optimal Evolution Points
  - Equidistant from simplex boundaries

$$a = \frac{1}{n + \sqrt{n}}$$



2nd-Order Generalization to R^n

- Reconstruct via polynomial

$$\varphi_\pm(x,t^m)=\varphi_\alpha+\sum_{k=1}^{n}\frac{\Delta_k^\pm\varphi_\alpha^m}{\Delta x}\left(x^{(k)}-x_\alpha^{(k)}\right)+\frac{1}{2}\sum_{k=1}^{n}\frac{\mathcal{D}_k\Delta_k^\pm\varphi_\alpha^m}{(\Delta x)^2}\left(x^{(k)}-x_\alpha^{(k)}\right)\left(x^{(k)}-x_{\alpha\pm e_k}^{(k)}\right)+\frac{1}{2}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n}\frac{\mathcal{D}_j\Delta_k^\pm\varphi_\alpha^m}{(\Delta x)^2}\left(x^{(j)}-x_\alpha^{(j)}\right)\left(x^{(k)}-x_\alpha^{(k)}\right)$$

  - Where $\mathcal{D}$ is the min-mod limited derivative

- At evolution points: at each point x_α

$$\varphi_\pm^m=\varphi_\alpha^m+a\sum_{k=1}^{n}\Delta_k^\pm\varphi_\alpha^m+\frac{a(a-1)}{2}\sum_{k=1}^{n}\mathcal{D}_k\Delta_k^\pm\varphi_\alpha^m+\frac{a^2}{2}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n}\mathcal{D}_j\Delta_k^\pm\varphi_\alpha^m$$

$$\left(\frac{\partial\varphi}{\partial x^{(p)}}\right)_\pm^m=\frac{\Delta_p^\pm\varphi_\alpha^m}{\Delta x}\pm\frac{2a-1}{2}\frac{\mathcal{D}_p\Delta_p^\pm\varphi_\alpha^m}{\Delta x}\pm\frac{a^2}{2}\sum_{\substack{k=1\\k\neq p}}^{n}\frac{\mathcal{D}_p\Delta_k^\pm\varphi_\alpha^m+\mathcal{D}_k\Delta_p^\pm\varphi_\alpha^m}{\Delta x}$$

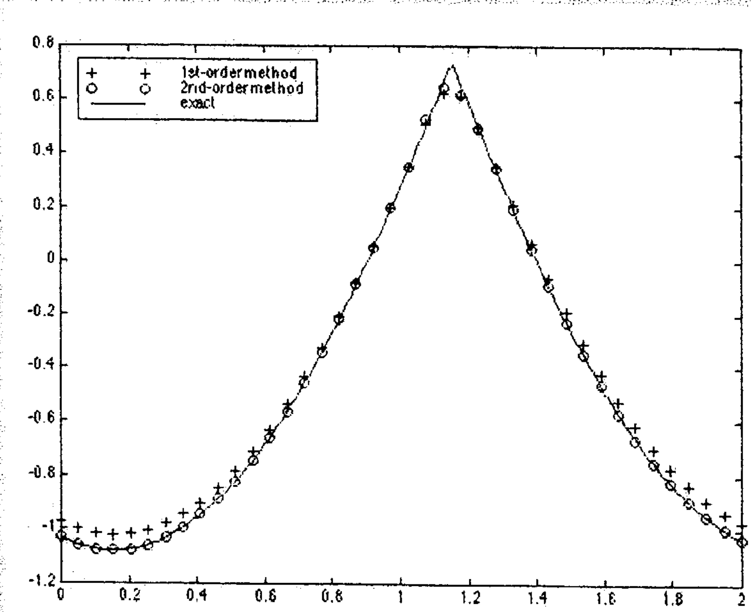
$$\left(\frac{\partial\varphi}{\partial x^{(p)}}\right)_\pm^{m+\frac{1}{2}}=\left(\frac{\partial\varphi}{\partial x^{(p)}}\right)_\pm^m-\frac{\Delta t}{2}\left[H_p\left((\nabla\varphi)_\pm^m\right)+\sum_{k=1}^{n}H_{\varphi_p}\left((\nabla\varphi)_\pm^m\right)\frac{\mathcal{D}_p\Delta_k^\pm\varphi_\alpha^m+\mathcal{D}_k\Delta_p^\pm\varphi_\alpha^m}{2(\Delta x)^2}\right]$$

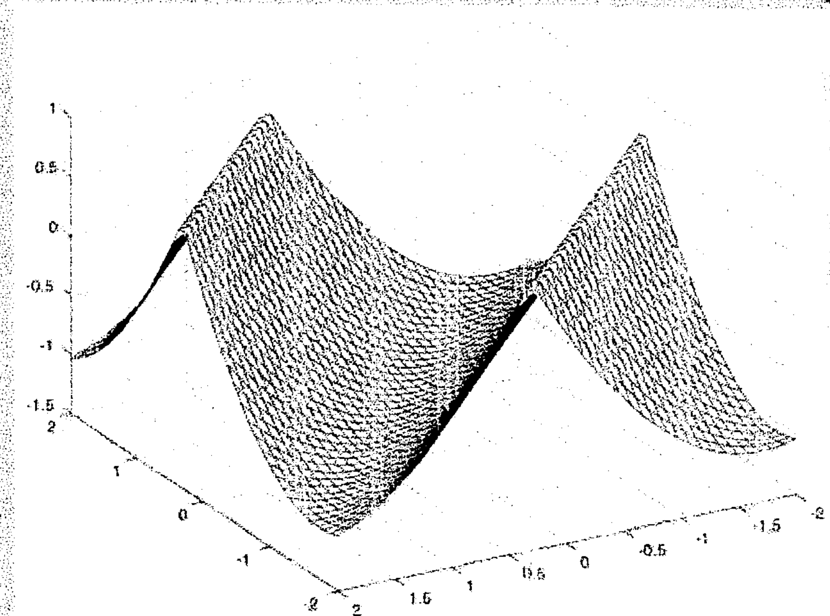
$$\varphi_\pm^{m+1}=\varphi_\pm^m-\Delta t H\left((\nabla\varphi)_\pm^{m+\frac{1}{2}}\right)\quad \text{where}\quad (\nabla\varphi)_\pm^m=\left(\left(\frac{\partial\varphi}{\partial x^{(1)}}\right)_\pm^m,\dots,\left(\frac{\partial\varphi}{\partial x^{(n)}}\right)_\pm^m\right)$$

- Reproject: $\varphi_\alpha^{m+1}=\frac{1}{2}\left(\varphi_+^m+\varphi_-^m\right)-a^2\sqrt{n}\Delta x\mathcal{D}(d\varphi)_0^{m+1}$

Convex H Example

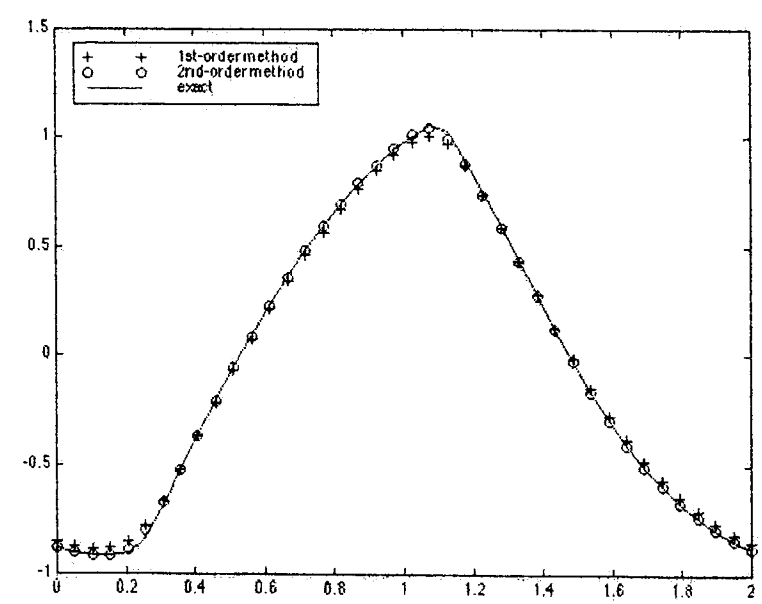
$$\begin{cases} \varphi_t + \frac{1}{2}(\varphi_x + 1)^2 = 0 \\ \varphi(x,0) = -\cos(\pi x) \end{cases}, \quad \begin{cases} \varphi_t + \frac{1}{2}(\varphi_x + \varphi_y + 1)^2 = 0 \\ \varphi(x,0) = -\cos\left(\frac{1}{2}\pi(x+y)\right) \end{cases}$$

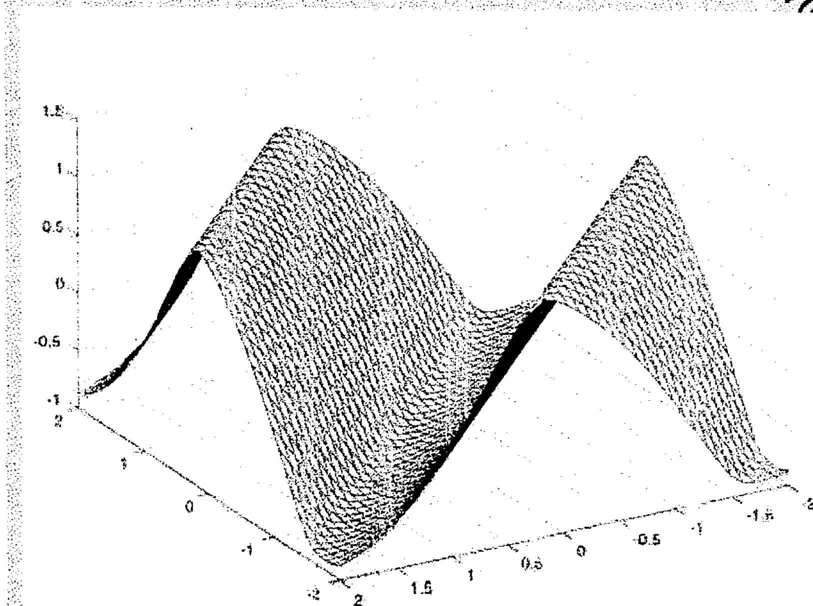




Non-Convex H Example

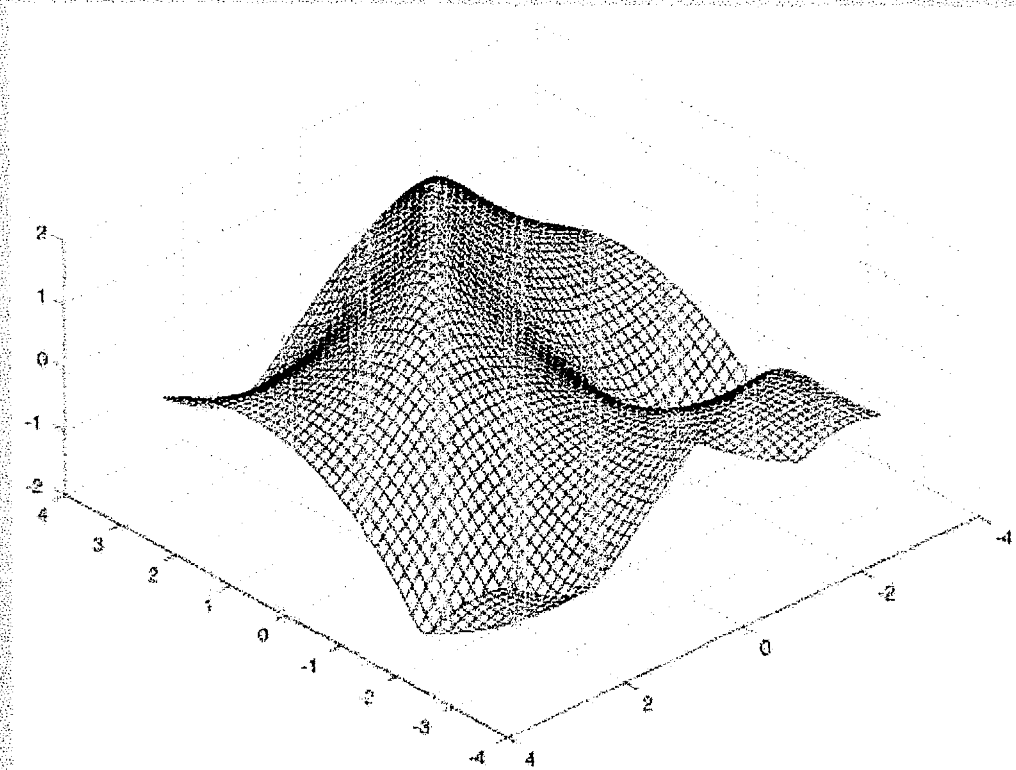
$$\begin{cases} \varphi_t - \cos(\varphi_x + 1) = 0 \\ \varphi(x,0) = -\cos(\pi x) \end{cases}, \quad \begin{cases} \varphi_t - \cos(\varphi_x + \varphi_y + 1) = 0 \\ \varphi(x,y,0) = -\cos\left(\frac{1}{2}\pi(x+y)\right) \end{cases}$$



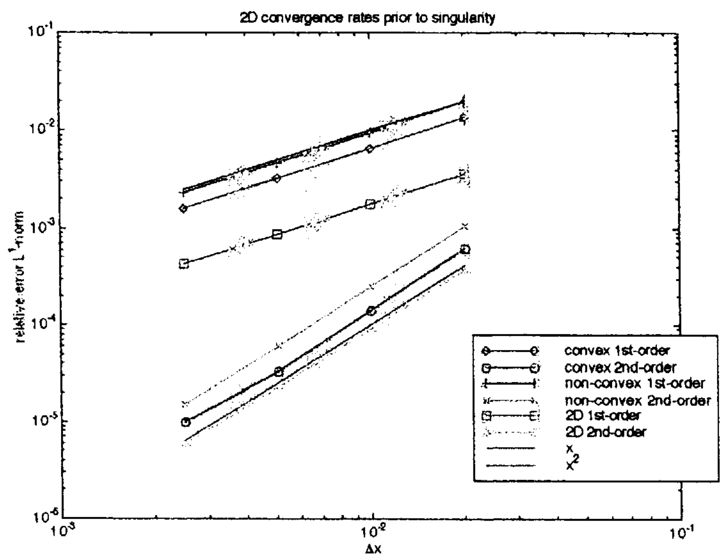


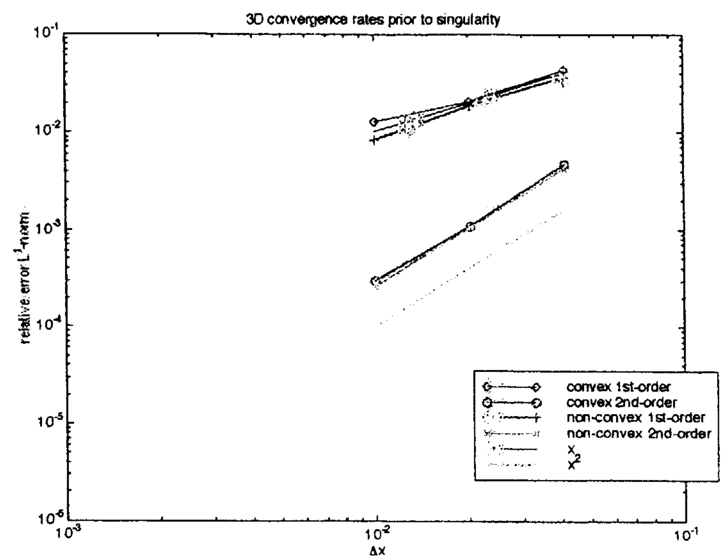
2D Example

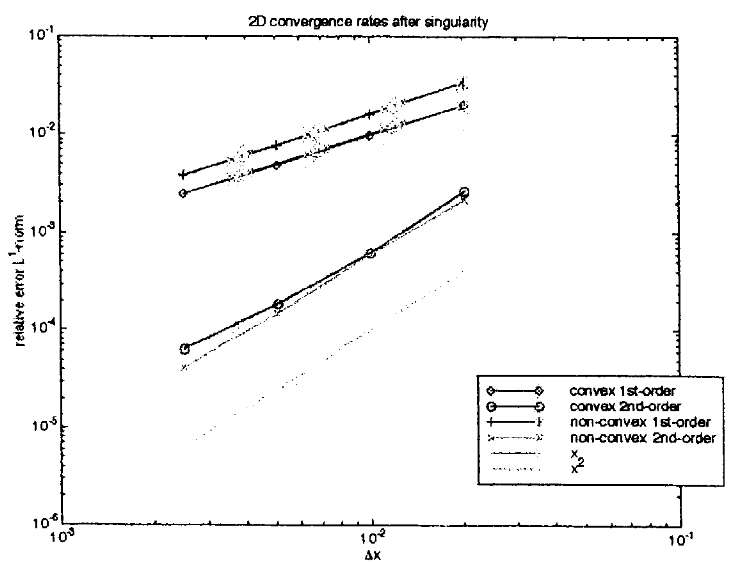
$$\begin{cases} \varphi_t + \varphi_x \varphi_y = 0 \\ \varphi(x, y, 0) = \sin(x) + \cos(y) \end{cases}$$

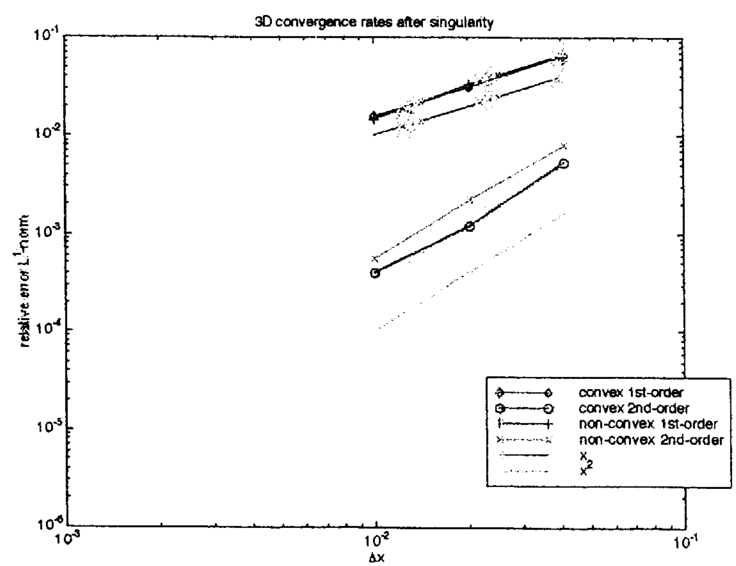


Convergence Rates









Higher Order

- Strategy:
  - Central WENO for reconstructions
  - Simpson's formula/SSP RK4 for evolution
    - Involves upwind WENO reconstruction of derivatives for each RK4 step

High-order 1D Interpolants

- 3rd-order example

$$\varphi_1(x_i + a\Delta x) = \left(-\frac{1}{2}a + \frac{1}{2}a^2\right)\varphi_{i-1} + \left(1 - a^2\right)\varphi_i + \left(\frac{1}{2}a + \frac{1}{2}a^2\right)\varphi_{i+1} = \varphi(x_i + ah) + O\left((\Delta x)^3\right)$$

$$\varphi_2(x_i + a\Delta x) = \left(1 - \frac{3}{2}a + \frac{1}{2}a^2\right)\varphi_i + \left(2a - a^2\right)\varphi_{i+1} + \left(-\frac{1}{2}a + \frac{1}{2}a^2\right)\varphi_{i+2} = \varphi(x_i + ah) + O\left((\Delta x)^3\right)$$

$$\varphi_c(x_i + a\Delta x) = c_1\varphi_1(x_i + a\Delta x) + c_2\varphi_2(x_i + a\Delta x) = \varphi(x_i + a\Delta x) + O\left((\Delta x)^4\right)$$

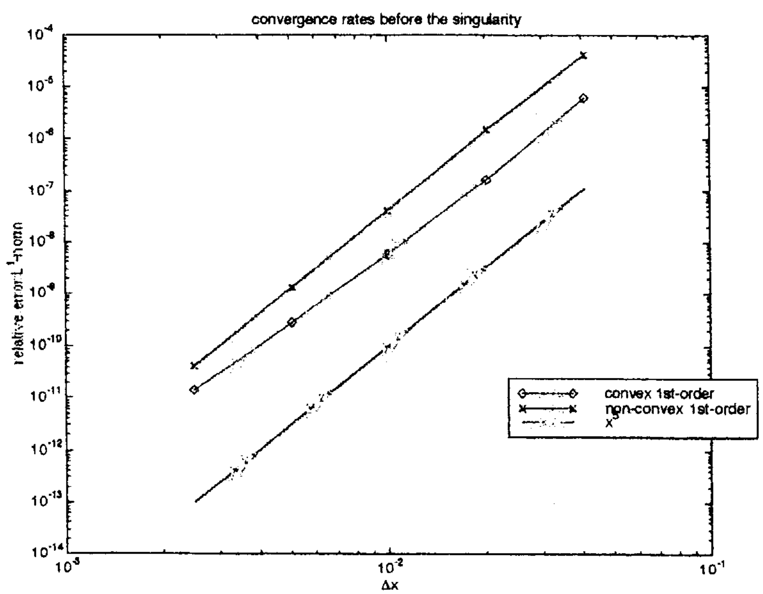
$$c_1 = \frac{1}{3}(2 - a),\ c_2 = \frac{1}{3}(1 + a)$$

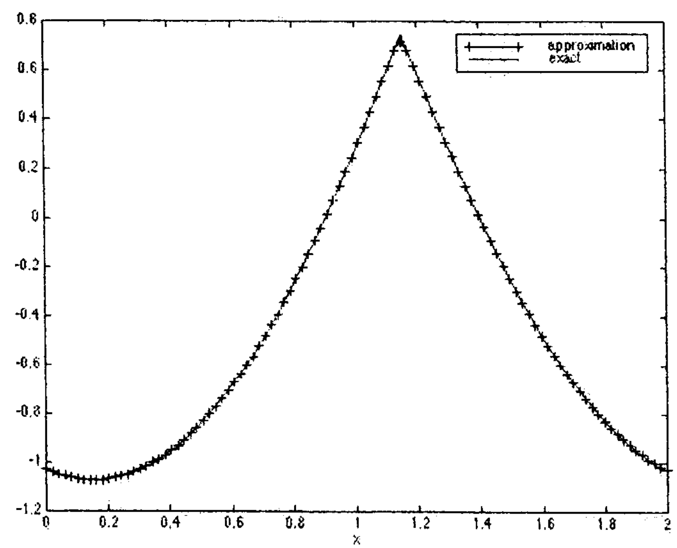
So set $\varphi_w^{\pm}(x_i \pm a\Delta x) = w_1\varphi_1^{\pm}(x_i \pm a\Delta x) + w_2\varphi_2^{\pm}(x_i \pm a\Delta x)$

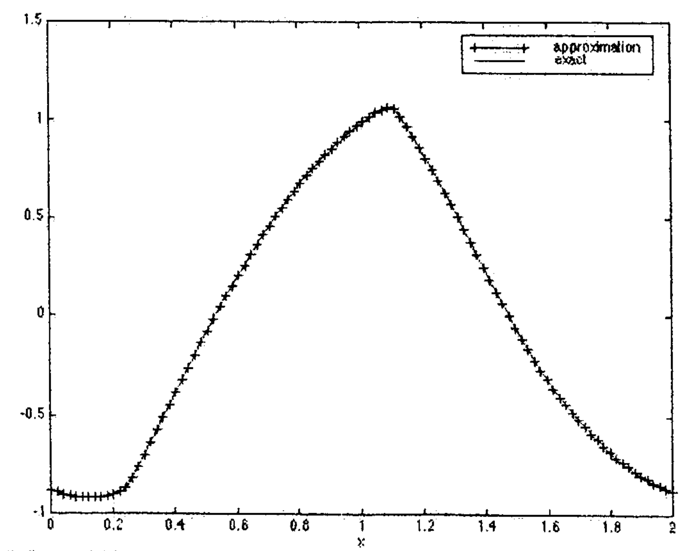
where $w_j = \dfrac{\alpha_j}{\alpha_1 + \alpha_2},\ \alpha_j = \dfrac{c_j}{(\varepsilon + S)^p}$ are defined

to attain high order in smooth regions while suppressing oscillatory interpolants

5th-order 1D Results





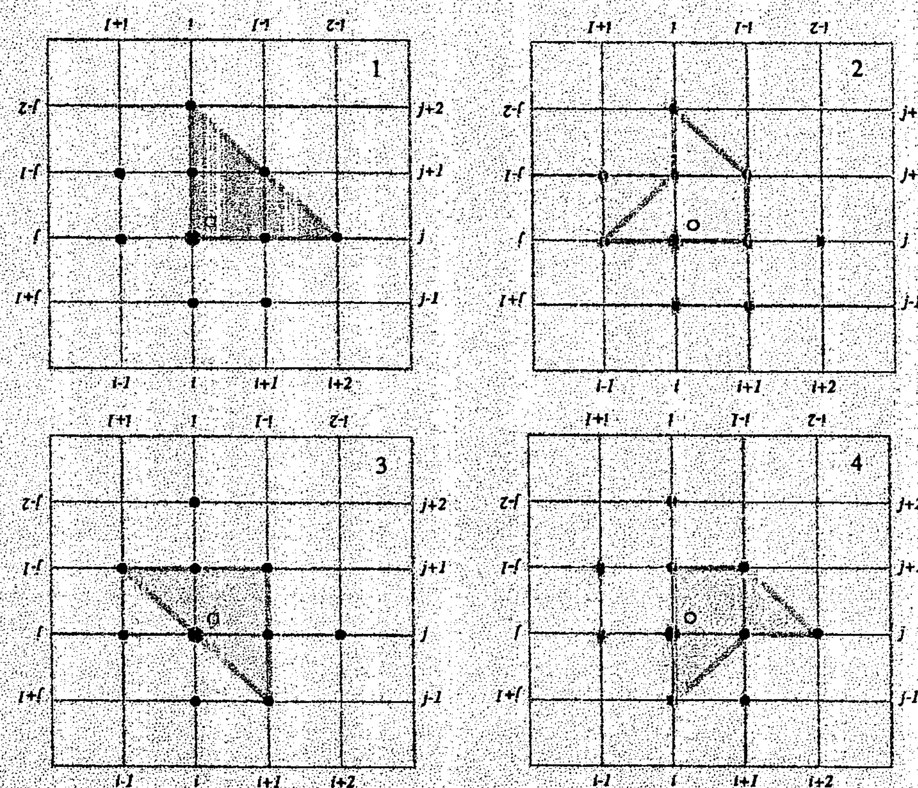


High-order 2D Reconstruction

- Three options for reconstruction
  - 2D interpolation
  - Direction-by-direction
  - Interpolate along diagonal
- In all cases, reconstruct derivatives via upwind interpolation

High-order 2D Stencils

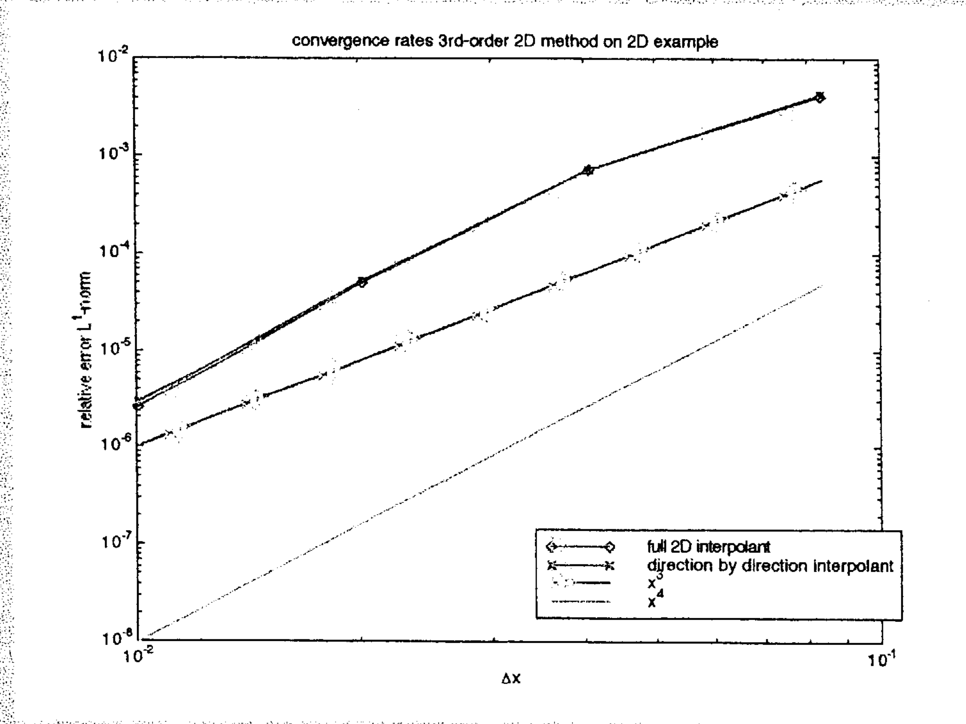
- 3rd-order example
- Stencils enclose evolution point
- Combination covers 10 points required for third order
- Use WENO combination to suppress stencils with oscillations



Direction-by-Direction Strategy

- In 2D:
  - 1: interpolate values along coordinate axes
  - 2: average coordinate interpolations to evolution point
- In *n*-D:
  - Iterate *n* steps, each with *n* interpolations

3rd-order Results

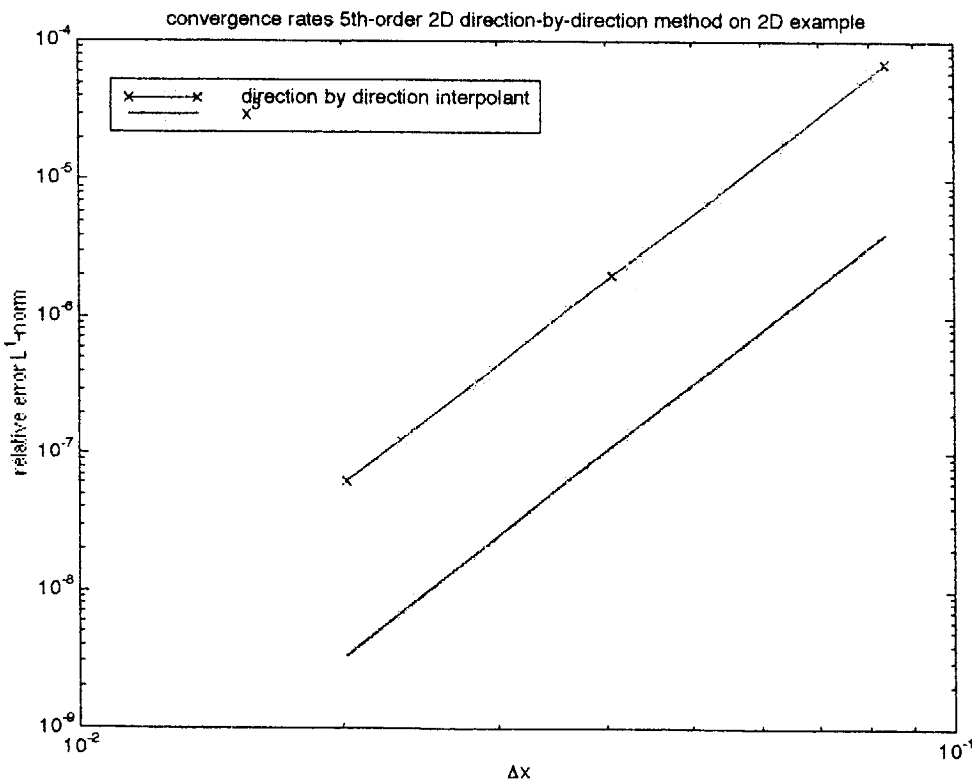


- Full 2D and direction by direction interpolation have similar quality

5th-order 2D

- Direction-by-direction CWENO reconstruction
- Upwind estimation of derivatives from Jiang and Peng
- Simpson's method for time evolution, using SSP RK4 for mid-values

5th-order 2D Results



Scaling to N Dimensions

- Direction by direction will scale better to high dimension than fully dimensional interpolation
- What about upwind? Requires estimation of the maximum of the gradient of H at each point
  - Significant computational burden

Conclusions

- Developed efficient high-order methods for HJ equations based on central methods
  - No need to estimate numerical Fluxes
- Scale well to high dimensions